ECS 315: Probability and Random Processes **2010/1**

EXAM 1 — Name ID

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) Conditions of Examination:

- Closed book
- Calculator (e.g. FX-991MS) allowed)

(b) Read these instructions and the questions carefully.

(c) Students are not allowed to be out of the examination room during examination. Going to the restroom may result in score deduction.

(d) Turn off all communication devices and place them with other personal belongings in the area designated by the proctors or outside the test room.

(e) Write your name, student ID, section, and seat number clearly in the spaces provided on the top of this sheet. Then, write your first name and the last three digits of your ID in the spaces provided on the top of each page of your examination paper, starting from page 2.

(f) The examination paper is not allowed to be taken out of the examination room. Violation may result in score deduction.

(g) Unless instructed otherwise, write down all the steps that you have done to obtain your answers.

- You may not get any credit even when your final answer is correct without showing how you get your answer.
- Exception: The 1-pt questions will be graded on your answers. For these questions, because there is no partial credit, it is not necessary to write down your explanation.

(h) When not explicitly stated/defined, all notations and definitions follow ones given in lecture.

(i) Some points are reserved for accuracy of the answers and also for reducing answers into their simplest forms.

(j) Points marked with * indicate challenging problems.

(k) Do not cheat. Do not panic. Allocate your time wisely.

Problem 1. (18 pt) In an experiment, A, B, C, and D are events with probabilities $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{8}$, $P(C) = \frac{5}{8}$, and $P(D) = \frac{3}{8}$. Furthermore, A and B are disjoint, while C and D are independent.

By defn, $A \cap B = \emptyset$

$A \perp B$

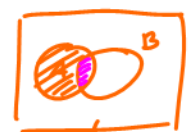
(a) Find

(i) (2 pt) $P(A \cap B)$ $= P(\emptyset) = 0$

Property proved in class.

(ii) (2 pt) $P(A \cup B)$ $= P(A) + P(B) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$

$A \perp B$



Alternatively,

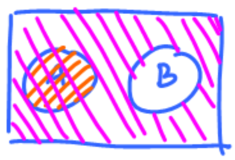
$P(A \cap B^c) = P(A \setminus B)$

$= P(A) - P(A \cap B)$

(a) $= P(A) - 0$

$= \frac{1}{4}$

(iii) (2 pt) $P(A \cap B^c)$



From Venn diagram,

$A \cap B^c = A$

$A \perp B$

$P(A \cap B^c) = P(A) = \frac{1}{4}$

(iv) (2 pt) $P(A \cup B^c)$

$A \cup B^c = B^c$

$P(A \cup B^c) = P(B^c) = 1 - P(B) = 1 - \frac{1}{8} = \frac{7}{8}$

$P(A \cup B^c) = P(A) + P(B^c) - P(A \cap B^c)$

$= \cancel{\frac{1}{4}} + \left(1 - \frac{1}{8}\right) - \cancel{\frac{1}{4}}$

$= \frac{7}{8}$

(b) (1 pt) Are A and B independent?

$P(A)P(B) \overset{?}{=} P(A \cap B)$ $\Rightarrow$ Not inpendent.

$\frac{1}{4} \times \frac{1}{8} \quad \times \quad 0$

(c) (2 pt) Note that $P(C) + P(D) = 1$. Does this mean $\underbrace{D = C^c}$? Justify your answer.

$\Downarrow$

No

① $D \perp C$

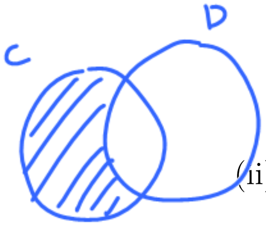
② $D \perp\!\!\!\perp C$

$P(\overbrace{D \cap C}^{\emptyset}) = P(D)P(C)$

$0 = P(D)P(C)$

(d) Find

(i) (2 pt) $P(C \cap D)$ $= P(C) \times P(D) = \frac{5}{8} \times \frac{3}{8} = \frac{15}{64}$



(ii) (2 pt) $P(C \cap D^c)$ $= P(C) - P(C \cap D) = \frac{5}{8} - \frac{15}{64} = \frac{25}{64}$

$C \perp\!\!\!\perp D \Leftrightarrow C \perp\!\!\!\perp D^c$ $\quad P(C \cap D^c) = P(C)P(D^c) = \frac{5}{8}\left(1 - \frac{3}{8}\right) = \frac{25}{64}$

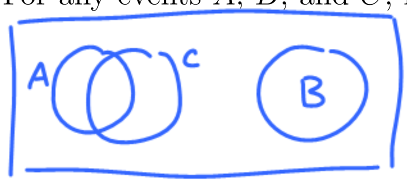
(iii) (2 pt) $P(C^c \cap D^c)$ $= P(C^c)P(D^c) = \left(1 - \frac{5}{8}\right)\left(1 - \frac{3}{8}\right) = \frac{15}{64}$

$C \perp\!\!\!\perp D \Leftrightarrow C^c \perp\!\!\!\perp D^c$

(e) (1 pt) Are C^c and D^c independent?

Problem 2. (10 pt) [M2010/1] Specify whether each of the following statements is TRUE or FALSE. If it is FALSE, provide your counter-example or explain why it is FALSE.

F (a) For any events A, B, and C, if $A \perp B$ and $B \perp C$, then $A \perp C$. False



F (b)* If $P(A \cup B) = P(A) + P(B)$, then A and B are disjoint. False

$P(A \cup B) = \underbrace{P(A) + P(B)}_{= P(A \cup B)} - P(A \cap B)$

$P(A \cap B) = 0$?

$A \cap B$ may not be empty but simply has elements whose probabilities are 0.

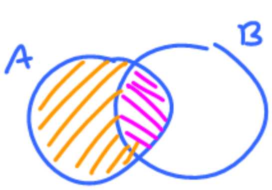
$P(E) = 0 \Rightarrow E = \emptyset$?

$E = \{a, b\}$ $\quad P(\{a\}) = 0$ $\quad P(\{b\}) = 0$

T (c) If $A \perp\!\!\!\perp B$, then $P(A) = P(A \cap B^c) + P(A)P(B)$.

$\| \qquad \perp\!\!\!\perp$

$P(A \cap B^c) + P(A \cap B)$



$\Omega = \{1, 2, 3\}$ $\quad P(\{1\}) = P(\{2\}) = \frac{1}{2}$ $\quad P(\{3\}) = 0$

$A = \{1, 3\}$ $\quad B = \{2, 3\}$

$P(A \cup B) = \frac{1}{2} + \frac{1}{2} + 0 = 1$ $\quad P(A) = \frac{1}{2} + 0 = \frac{1}{2}$ $\quad P(B) = \frac{1}{2} + 0 = \frac{1}{2}$

F (d) For any events A, B, and C, if $A \perp\!\!\!\perp B$ and $B \perp\!\!\!\perp C$, then $A \perp\!\!\!\perp C$.

$B = \Omega$ $\qquad P(A\cancel{\cap B}) = P(A)\cancel{P(B)}$

F (e) For any events A, B, and C, if $A \perp\!\!\!\perp B$, $B \perp\!\!\!\perp C$, and $A \perp\!\!\!\perp C$, then the events A, B, and C are independent.

Need to have $P(A \cap B \cap C) = P(A)P(B)P(C)$

Problem 3. (12 pt) [M2010/1] Roll a fair six-sided dice five times. Let X_i be the number of dots that show up on the ith roll.

(a) (4 pt) List all $(X_1, X_2, X_3, X_4, X_5)$ where $X_i \in \{1, 2, 3, 4, 5, 6\}$ such that $X_1 + X_2 + X_3 + X_4 + X_5 = 6$. There should be 5 of these.

X_1	X_2	X_3	X_4	X_5
1	1	1	1	2
1	1	1	2	1
1	1	2	1	1
1	2	1	1	1
2	1	1	1	1

(b) (4 pt) What is the probability that $X_1 + X_2 + X_3 + X_4 + X_5 = 6$?

$$\frac{5}{6^5} = \frac{5}{7776} \approx 6.43 \times 10^{-4}$$

(c) (2 pt) What is the probability that $X_1 + X_2 + X_3 + X_4 + X_5 = 10$?

$$\frac{\binom{9}{5}}{6^5}$$

(d) (2 pt) Given that $X_1 + X_2 + X_3 + X_4 + X_5 = 6$, find the probability that $X_1 = 1$.

$\frac{4}{5}$

Problem 4. (6 pt) Suppose that for the Country of Oz, 1 in 1000 people carries the human immunodeficiency virus (HIV). A test for the presence of HIV yields either a positive (+) or negative (-) response. Suppose the test gives the correct answer 95% of the time. We would like to find the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive.

(a) (2 pt) What is $P(-|H)$, the conditional probability that a person tests negative given that the person does have the HIV virus?

$P(-|H) = 1 - P(+|H) = 1 - 0.95 = 0.05$

(b) (2 pt) ***Use*** the law of total probability to find $P(+)$, the probability that a randomly chosen person tests positive. Provide at least 3 significant digits in your answer.

$= P(+ \cap H) + P(+ \cap H^c)$

$P(+) = P(+|H)P(H) + P(+|H^c)P(H^c)$

$= 0.95 \times \frac{1}{1000} + 0.05 \times \left(1 - \frac{1}{1000}\right)$

$= \ldots$

(c) (2 pt) ***Use*** Bayes' formula to find $P(H|+)$, the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive. Provide at least 3 significant digits in your answer.

$P(H|+) = \frac{P(+|H)P(H)}{P(+)} = \frac{0.95 \times \frac{1}{1000}}{} = \ldots$

Problem 5. (33 pt) The random variable V has pmf

$$p_V(v) = \begin{cases} \frac{1}{v^2} + c, & v \in \{-2, 2, 3\} \\ 0, & \text{otherwise.} \end{cases}$$

(a) (5 pt) Find the value of the constant c.

$$\sum_v p_V(v) = 1 \Rightarrow \left(\frac{1}{(-2)^2}+c\right)+\left(\frac{1}{2^2}+c\right)+\left(\frac{1}{3^2}+c\right) = 1$$

$$\frac{1}{4}+\frac{1}{4}+\frac{1}{9}+3c = 1$$

$$\frac{11}{18}+3c = 1$$

$$3c = \frac{7}{18}$$

$$c = \frac{7}{54}$$

(b) (2 pt) Find $P[V > 3]$. $= 0$

All possible values of V are ≤ 3

(c) (2 pt) Find $P[V < 3]$. $= p_V(-2) + p_V(2) = \frac{1}{(-2)^2} + c + \frac{1}{2^2} + c = \frac{1}{4} + \frac{1}{4} + 2c$

v	<3?
-2	✓
2	✓
3	×

$\rightarrow$ V can be $= -2$ or 2

$$= \frac{1}{2} + 2c = \frac{1}{2} + \frac{14}{54} = \frac{41}{54}$$

(d) (2 pt) Find $P[V^2 > 1]$. $= 1$

v	v^2	>1?
-2	4	✓
2	4	✓
3	9	✓

(e) (3 pt) Sketch $p_V(v)$. Provide as much information on the sketch as you can.

(f) (3 pt) Sketch $F_V(v)$. Provide as much information on the sketch as you can.

(g) (4 pt) Let $W = V^2 - V + 1$. Find the pmf of W.

(h) (3 pt) Find $\mathbb{E}V$

(i) (3 pt) Find $\mathbb{E}[V^2]$

(j) (3 pt) Find $\operatorname{Var} V$

(k) (1 pt) Find σ_V

(l) (2 pt) Find $\mathbb{E}W$

Problem 6. (16 pt) The input X and output Y of a system subject to random perturbations are described probabilistically by the following joint pmf matrix:

$$\begin{array}{c|c} x \backslash y & \begin{matrix} 2 & \;\;\; 4 & \;\;\; 5 \end{matrix} \\ \begin{matrix} 1 \\ 3 \end{matrix} & \begin{bmatrix} 0.02 & 0.10 & 0.08 \\ 0.08 & 0.32 & 0.40 \end{bmatrix} \end{array}$$

(a) (2 pt) Find the marginal pmf $p_X(x)$.

(b) (2 pt) Find the marginal pmf $p_Y(y)$.

(c) (2 pt) Find $\mathbb{E}X$

(d) (2 pt) Find $P[X = Y]$

(e) (2 pt) Find $P[XY < 6]$

(f) (2 pt) Find $\mathbb{E}\left[(X-3)(Y-2)\right]$

(g) (2 pt) Find $\mathbb{E}\left[X(Y^3 - 11Y^2 + 38Y)\right]$

(h) (2 pt) Are X and Y independent?

Problem 7. (2 pt) A random variables X has support containing only two numbers. Its expected value is $\mathbb{E}X = 5$. Its variance is $\operatorname{Var} X = 3$. Give an example of the pmf of such a random variable.

Problem 8. (2 pt) [M2010/1] Suppose $X_1 \sim \text{Bernoulli}(1/3)$ and $X_2 \sim \text{Bernoulli}(1/4)$. Assume that $X_1 \perp\!\!\!\perp X_2$.

(a) (1 pt) Find the joint pmf matrix of the pair (X_1, X_2).

(b) (1 pt) Find the pmf of $Y = X_1 + X_2$.

Problem 9. (1 pt) [M2010/1] Suppose X and Y are i.i.d. random variables. Suppose $\operatorname{Var} X = 5$ Find $\mathbb{E}\left[(X-Y)^2\right]$.